

Hyperbolic Cauchy Integral Formula for the Split Complex Numbers

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Abstract

In our joint papers [FL1, FL2] we revive quaternionic analysis and show deep relations between quaternionic analysis, representation theory and four-dimensional physics. As a guiding principle we use representation theory of various real forms of the conformal group. We demonstrate that the requirement of unitarity of representations naturally leads us to the extensions of the Cauchy-Fueter and Poisson formulas to the Minkowski space, which can be viewed as another real form of quaternions. However, the Minkowski space formulation also brings some technical difficulties related to the fact that the singularities of the kernels in these integral formulas are now concentrated on the light cone instead of just a single point in the initial quaternionic picture. But the same phenomenon occurs when one passes from the complex numbers to the split complex numbers (or hyperbolic algebra). So, as a warm-up example we proved an analogue of the Cauchy integral formula for the split complex numbers. On the other hand, there seems to be sufficient interest in such formula among physicists. For example, see [KS] and the references therein.

In this short article we give a Cauchy-type integral formula for solutions of the wave equation $\square_{1,1}F = 0$ on $\mathbb{R}^2 = \{(x, y); x, y \in \mathbb{R}\}$, where

$$\square_{1,1}F = \frac{\partial^2}{\partial x^2}F - \frac{\partial^2}{\partial y^2}F.$$

More precisely, we write elements (x, y) of \mathbb{R}^2 as $Z = x + jy$ and equip \mathbb{R}^2 with multiplication operation so that $j^2 = 1$. Then \mathbb{R}^2 becomes an algebra over \mathbb{R} , and we denote this algebra by $\mathbb{R}^{1,1}$. We introduce two differential operators

$$\partial_{1,1} = \frac{1}{2}\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) \quad \text{and} \quad \partial_{1,1}^+ = \frac{1}{2}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right).$$

Note that

$$\partial_{1,1}\partial_{1,1}^+ = \partial_{1,1}^+\partial_{1,1} = \frac{1}{4}\square_{1,1}.$$

We will give an integral formula for differentiable functions $F : \mathbb{R}^{1,1} \rightarrow \mathbb{R}^{1,1}$ such that $\partial_{1,1}^+F = 0$.

Proposition 1 *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be a pair of smooth single-variable functions. Then*

$$F(x, y) = (f(x + y) + jf(x - y)) + (g(x - y) - jg(x + y)) \quad (1)$$

is a solution of $\partial_{1,1}^+F = 0$. Moreover, any smooth solution of $\partial_{1,1}^+F = 0$ is of this form.

Proof. The first statement is straightforward. To prove the second statement, let F be such that $\partial_{1,1}^+ F = 0$. Write

$$F = \frac{1+j}{2} \cdot F + \frac{1-j}{2} \cdot F.$$

Then

$$0 = \frac{1+j}{2} \cdot \partial_{1,1}^+ F = \frac{1+j}{2} \cdot \frac{1}{2} \left(\frac{\partial}{\partial x} - j \frac{\partial}{\partial y} \right) F = \frac{1}{2} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \left(\frac{1+j}{2} \cdot F \right)$$

which implies that $\frac{1+j}{2} \cdot F$ can be written as $f(x+y) + jf(x+y)$ for some smooth single-variable function $f : \mathbb{R} \rightarrow \mathbb{R}$. Similarly,

$$0 = \frac{1-j}{2} \cdot \partial_{1,1}^+ F = \frac{1-j}{2} \cdot \frac{1}{2} \left(\frac{\partial}{\partial x} - j \frac{\partial}{\partial y} \right) F = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left(\frac{1-j}{2} \cdot F \right)$$

which implies that $\frac{1-j}{2} \cdot F$ can be written as $g(x-y) - jg(x-y)$ for some smooth single-variable function $g : \mathbb{R} \rightarrow \mathbb{R}$. \square

Corollary 2 *Let $F : \mathbb{R}^{1,1} \rightarrow \mathbb{R}^{1,1}$ be a smooth function satisfying $\partial_{1,1}^+ F = 0$, then $\frac{1+j}{2} \cdot F$ is constant along the lines $x+y = \text{const}$ and $\frac{1-j}{2} \cdot F$ is constant along the lines $x-y = \text{const}$.*

Lemma 3 *If $F, G : \mathbb{R}^{1,1} \rightarrow \mathbb{R}^{1,1}$ are smooth functions satisfying $\partial_{1,1}^+ F = \partial_{1,1}^+ G = 0$, then so is their product FG .*

Proof. $\partial_{1,1}^+(FG) = (\partial_{1,1}^+ F)G + F(\partial_{1,1}^+ G) = 0$. \square

Lemma 4 *Let $dZ = dx + jdy$, and let $F : \mathbb{R}^{1,1} \rightarrow \mathbb{R}^{1,1}$ be a differentiable function. Then*

$$d(F dZ) = 2j \cdot (\partial_{1,1}^+ F) dx \wedge dy.$$

In particular,

$$d(F dZ) = 0 \quad \Longleftrightarrow \quad \partial_{1,1}^+ F = 0.$$

Let $U \subset \mathbb{R}^{1,1}$ be an open subset with piecewise smooth boundary ∂U . We give a canonical orientation to ∂U as follows. The positive orientation of U is determined by $\{1, j\}$. Pick a smooth point $p \in \partial U$ and let \vec{n}_p be a non-zero vector in $T_p \mathbb{R}^{1,1}$ perpendicular to $T_p \partial U$ and pointing outside of U . We give ∂U an orientation so that a tangent vector $\vec{\tau}_p \neq 0$ at $p \in \partial U$ points in the positive direction if and only if $\{\vec{n}_p, \vec{\tau}_p\}$ in $\mathbb{R}^{1,1}$.

Corollary 5 *Let $U \subset \mathbb{R}^{1,1}$ be an open bounded subset as above, let V be an open neighborhood of the closure \overline{U} . And let $F : V \rightarrow \mathbb{R}^{1,1}$ be a smooth function such that $\partial_{1,1}^+ F = 0$. Then*

$$\int_{\partial U} F dZ = 0.$$

For $Z = x + jy$ set

$$Z^+ = x - jy, \quad N(Z) = x^2 - y^2, \quad \|Z\| = \sqrt{x^2 + y^2}.$$

Lemma 6 *1. An element $Z = x + jy \in \mathbb{R}^{1,1}$ is invertible if and only if $N(Z) = x^2 - y^2 \neq 0$; in which case $Z^{-1} = \frac{Z^+}{N^2(Z)}$;*

2. If $Z, W \in \mathbb{R}^{1,1}$, then $N(ZW) = N(Z) \cdot N(W)$;

3. The function $K(Z) = Z^{-1} = \frac{Z^+}{N^2(Z)}$ satisfies $\partial_{1,1}^+ K = 0$ wherever $N(Z) \neq 0$.

Observe that

$$\begin{aligned} K(x + jy) &= \frac{x - jy}{x^2 - y^2} = \frac{1+j}{2} \cdot \frac{x - jy}{x^2 - y^2} + \frac{1-j}{2} \cdot \frac{x - jy}{x^2 - y^2} \\ &= \frac{1+j}{2} \cdot \frac{x - y}{x^2 - y^2} + \frac{1-j}{2} \cdot \frac{x + y}{x^2 - y^2} = \frac{1+j}{2} \cdot \frac{1}{x+y} + \frac{1-j}{2} \cdot \frac{1}{x-y}. \end{aligned} \quad (2)$$

Let $(\mathbb{R}^{1,1})^\times = \{x + jy \in \mathbb{R}^{1,1}; x^2 - y^2 \neq 0\}$ be the set of invertible elements. For $\varepsilon \in \mathbb{R}$, define $K_\varepsilon : (\mathbb{R}^{1,1})^\times \rightarrow \mathbb{R}^{1,1} \otimes_{\mathbb{R}} \mathbb{C}$ by:

$$K_\varepsilon(x + jy) = \frac{1+j}{2} \cdot \frac{1}{x + y + i\varepsilon \cdot \text{sign}(x - y)} + \frac{1-j}{2} \cdot \frac{1}{x - y + i\varepsilon \cdot \text{sign}(x + y)}.$$

Then on each connected component of $(\mathbb{R}^{1,1})^\times$ the function K_ε is of the type (1), hence satisfies $\partial_{1,1}^+ K_\varepsilon = 0$.

Proposition 7 (Integral Formula) *Let $R > 0$, set $U = \{Z \in \mathbb{R}^{1,1}; |N(Z)| < R\}$ and let V be an open neighborhood of the closure \bar{U} . Suppose $F : V \rightarrow \mathbb{R}^{1,1}$ is a smooth bounded function satisfying $\partial_{1,1}^+ F = 0$ and pick any $Z_0 = x_0 + jy_0 \in U$, then*

$$f(Z_0) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_{\{|N(Z)|=R\}} K_{Z_0,\varepsilon}(Z) \cdot F(Z) dZ, \quad (3)$$

where

$$K_{Z_0,\varepsilon} = \frac{1+j}{2} \cdot \frac{1}{x - x_0 + y - y_0 + i\varepsilon \cdot \text{sign}(x - y)} + \frac{1-j}{2} \cdot \frac{1}{x - x_0 - y + y_0 + i\varepsilon \cdot \text{sign}(x + y)},$$

the hyperbolas $N(Z) = \pm R$ are oriented as ∂U (i.e. counterclockwise) and the improper integral is defined as

$$\int_{\{|N(Z)|=R\}} K_{Z_0,\varepsilon}(Z) \cdot F(Z) dZ = \lim_{S \rightarrow \infty} \int_{\{|N(Z)|=R\} \cap \{\|Z\| \leq S\}} K_{Z_0,\varepsilon}(Z) \cdot F(Z) dZ.$$

Note that $K_{Z_0,0} = K(Z - Z_0)$, so we can regard $K_{Z_0,\varepsilon}(Z)$ as a perturbation of $K(Z - Z_0) = \frac{1}{Z - Z_0}$. Thus the integral formula formally looks identical to the Cauchy integral formula for holomorphic functions.

Proof. Write $K_{Z_0,\varepsilon}$ as $K_{Z_0,\varepsilon}^+ + K_{Z_0,\varepsilon}^-$, where

$$\begin{aligned} K_{Z_0,\varepsilon}^+(Z) &= \frac{1+j}{2} \cdot \frac{1}{x - x_0 + y - y_0 + i\varepsilon \cdot \text{sign}(x - y)} \\ \text{and} \quad K_{Z_0,\varepsilon}^-(Z) &= \frac{1+j}{2} \cdot \frac{1}{x - x_0 - y + y_0 + i\varepsilon \cdot \text{sign}(x + y)}. \end{aligned}$$

To prove (3) it is enough to show that

$$2\pi i \cdot \frac{1 \pm j}{2} \cdot F(Z_0) = \lim_{\varepsilon \rightarrow 0^+} \left(\lim_{S \rightarrow \infty} \int_{\{|N(Z)|=R\} \cap \{\|Z\| \leq S\}} K_{Z_0,\varepsilon}^\pm(Z) \cdot F(Z) dZ \right). \quad (4)$$

Note that both $K_{Z_0, \varepsilon}^{\pm}$ satisfy $\partial_{1,1}^+ K_{Z_0, \varepsilon}^{\pm} = 0$ and that the integrand is a closed form. We also observe that

$$\begin{aligned} (1+j) dZ &= (1+j)(dx + jdy) = (1+j)d(x+y) \\ \text{and} \quad (1-j) dZ &= (1-j)(dx + jdy) = (1-j)d(x-y). \end{aligned}$$

We change coordinates to (u, v) so that

$$u = x + y, \quad v = x - y, \quad u_0 = x_0 + y_0, \quad v_0 = x_0 - y_0.$$

Then

$$\begin{aligned} K_{Z_0, \varepsilon}^+(Z) \cdot F(Z) dZ &= \frac{1+j}{2} \cdot \frac{F(u, v)}{u - u_0 + i\varepsilon \cdot \text{sign}(v)} du, \\ K_{Z_0, \varepsilon}^-(Z) \cdot F(Z) dZ &= \frac{1-j}{2} \cdot \frac{F(u, v)}{v - v_0 + i\varepsilon \cdot \text{sign}(u)} dv. \end{aligned}$$

The hyperbolas $\{N(Z) = \pm R\}$ in these coordinates become $\{uv = \pm R\}$.

Let

$$U_S = U \cap \{|u|, |v| < S\} = \{Z \in \mathbb{R}^{1,1}; -R < N(Z) < R \text{ and } |u|, |v| < S\},$$

and orient its boundary ∂U_S as in Corollary 5, i.e. counterclockwise.

Lemma 8

$$\lim_{S \rightarrow \infty} \left(\int_{\{|N(Z)|=R\} \cap \{|u|, |v| \leq S\}} K_{Z_0, \varepsilon}^{\pm}(Z) \cdot F(Z) dZ - \int_{\partial U_S} K_{Z_0, \varepsilon}^{\pm}(Z) \cdot F(Z) dZ \right) = 0$$

Proof. The difference of integrals in question is integral of $K_{Z_0, \varepsilon}^{\pm}(Z) \cdot F(Z) dZ$ over the four straight segments of the boundary ∂U_S :

$$\{Z \in \mathbb{R}^{1,1}; |v| = S \text{ and } |u| \leq R/S\} \quad \text{and} \quad \{Z \in \mathbb{R}^{1,1}; |u| = S \text{ and } |v| \leq R/S\}.$$

The length of each segment is $2R/S$. Since F is bounded and $|K_{Z_0, \varepsilon}^{\pm}| \leq \frac{1}{\varepsilon}$, the integrals over these segments tend to zero as $S \rightarrow \infty$. \square

Since the form $K_{Z_0, \varepsilon}^+(Z) \cdot F(Z) dZ$ is closed, by Stokes' theorem applied to the two regions $U_S \cap \{v > 0\}$ and $U_S \cap \{v < 0\}$ we have

$$\begin{aligned} \int_{\partial U_S} K_{Z_0, \varepsilon}^+(Z) \cdot F(Z) dZ &= \frac{1+j}{2} \int_{u=-S}^{u=S} \left(\frac{F(u, v)}{u - u_0 - i\varepsilon} \Big|_{v=0} - \frac{F(u, v)}{u - u_0 + i\varepsilon} \Big|_{v=0} \right) du \\ &= \frac{1+j}{2} \int_{u=-S}^{u=S} \frac{2i\varepsilon}{(u - u_0)^2 + \varepsilon^2} \cdot F(u, 0) du. \end{aligned}$$

As $S \rightarrow \infty$, we get

$$\lim_{S \rightarrow \infty} \int_{\partial U_S} K_{Z_0, \varepsilon}^+(Z) \cdot F(Z) dZ = \frac{1+j}{2} \int_{-\infty}^{\infty} \frac{2i\varepsilon}{(u - u_0)^2 + \varepsilon^2} \cdot F(u, 0) du.$$

Finally,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{2i\varepsilon}{(u-u_0)^2 + \varepsilon^2} \cdot F(u, 0) du &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{2i}{\left(\frac{u-u_0}{\varepsilon}\right)^2 + 1} \cdot F(u, 0) d\left(\frac{u-u_0}{\varepsilon}\right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{2i}{t^2 + 1} \cdot F(u_0 + \varepsilon t, 0) dt = 2\pi i \cdot F(u_0, 0). \end{aligned}$$

By Corollary 2, $\frac{1+j}{2} \cdot F(u_0, 0) = \frac{1+j}{2} \cdot F(u_0, v_0)$. This proves (4) for $K_{Z_0, \varepsilon}^+(Z)$.

Similarly, the form $K_{Z_0, \varepsilon}^-(Z) \cdot F(Z) dZ$ is closed, by Stokes' theorem applied to the two regions $U_S \cap \{u > 0\}$ and $U_S \cap \{u < 0\}$ we have

$$\begin{aligned} \int_{\partial U_S} K_{Z_0, \varepsilon}^-(Z) \cdot F(Z) dZ \\ &= \frac{1-j}{2} \int_{v=-S}^{v=S} \left(\frac{F(u, v)}{v-v_0-i\varepsilon} \Big|_{u=0} - \frac{F(u, v)}{v-v_0+i\varepsilon} \Big|_{u=0} \right) dv \\ &= \frac{1-j}{2} \int_{u=-S}^{u=S} \frac{2i\varepsilon}{(v-v_0)^2 + \varepsilon^2} \cdot F(0, v) dv. \end{aligned}$$

As $S \rightarrow \infty$, we get

$$\lim_{S \rightarrow \infty} \int_{\partial U_S} K_{Z_0, \varepsilon}^-(Z) \cdot F(Z) dZ = \frac{1-j}{2} \int_{-\infty}^{\infty} \frac{2i\varepsilon}{(v-v_0)^2 + \varepsilon^2} \cdot F(0, v) dv.$$

Finally,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{2i\varepsilon}{(v-v_0)^2 + \varepsilon^2} \cdot F(0, v) dv &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{2i}{\left(\frac{v-v_0}{\varepsilon}\right)^2 + 1} \cdot F(0, v) d\left(\frac{v-v_0}{\varepsilon}\right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{2i}{t^2 + 1} \cdot F(0, v_0 + \varepsilon t) dt = 2\pi i \cdot F(0, v_0). \end{aligned}$$

By Corollary 2, $\frac{1-j}{2} \cdot F(0, v_0) = \frac{1-j}{2} \cdot F(u_0, v_0)$. This proves the second part of (4). \square

Next we show how the requirement in the integral formula that F is bounded can be dropped.

Corollary 9 *As before, let $R > 0$, set $U = \{Z \in \mathbb{R}^{1,1}; |N(Z)| < R\}$ and let V be an open neighborhood of the closure \overline{U} . Suppose $F : V \rightarrow \mathbb{R}^{1,1}$ is a smooth function satisfying $\partial_{1,1}^+ F = 0$ and pick any $Z_0 = x_0 + jy_0 \in U$. Let $\varphi : \mathbb{R} \rightarrow [0, 1]$ be any smooth function with compact support such that $\varphi(0) = 1$. Then*

$$f(Z_0) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_{\{|N(Z)|=R\}} K_{Z_0, \varphi, \varepsilon}(Z) \cdot F(Z) dZ,$$

where

$$K_{Z_0, \varphi, \varepsilon} = \frac{1+j}{2} \cdot \frac{\varphi(x-x_0+y-y_0)}{x-x_0+y-y_0+i\varepsilon \cdot \text{sign}(x-y)} + \frac{1-j}{2} \cdot \frac{\varphi(x-x_0-y+y_0)}{x-x_0-y+y_0+i\varepsilon \cdot \text{sign}(x+y)}.$$

Proof. The functions

$$\begin{aligned} F_1(Z) &= \frac{1+j}{2} \cdot \varphi(x-x_0+y-y_0) \cdot F(Z-Z_0) \\ \text{and} \quad F_2(Z) &= \frac{1-j}{2} \cdot \varphi(x-x_0-y+y_0) \cdot F(Z-Z_0) \end{aligned}$$

are bounded, satisfy $\partial_{1,1}^+ F_1 = \partial_{1,1}^+ F_2 = 0$ and

$$F_1(Z_0) = \frac{1+j}{2} \cdot F(Z_0), \quad F_2(Z_0) = \frac{1-j}{2} \cdot F(Z_0).$$

Hence the integral formula (3) applies and the result follows. \square

References

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